NOTES

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A Theorem of Burnside on Matrix Rings

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1. BURNSIDE'S THEOREM. In 1905, Burnside [**2**] proved the following remarkable result on groups of invertible matrices over the complex field $\mathbb{C}$:

Theorem 1. *Let G be a group of invertible $n \times n$ matrices over $\mathbb{C}$. Then G has no nontrivial invariant subspaces in $\mathbb{C}^n$ if and only if G contains n^2 linearly independent matrices, that is, if and only if the $\mathbb{C}$-span of G in $\mathbf{M}_n(\mathbb{C})$ is $\mathbf{M}_n(\mathbb{C})$ itself.*

The "if" part is easy, since $\mathbf{M}_n(\mathbb{C})$ has no nontrivial invariant subspaces in $\mathbb{C}^n$ (the "trivial" ones being $\{0\}$ and $\mathbb{C}^n$). Thus, the gist of Burnside's Theorem is in its "only if" part.

For an explicit example, take G to be the dihedral group G generated by the rotation $r = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and the reflection $s = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. It can be seen that G has no invariant subspaces in $\mathbb{C}^2$, and in fact, r, s, $rs = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, together with the identity matrix clearly form a basis of $\mathbf{M}_2(\mathbb{C})$.

Burnside's Theorem (and its subsequent generalization by Frobenius and Schur in [**5**]) proved to be a fundamental result in the representation theory of groups, and has appeared in many books on that subject. From a ring-theoretic perspective, [**2**] and [**5**] yield a more general result, nowadays also called Burnside's Theorem, which can be formulated as follows.

Theorem 2. *Let A be a subring of $\mathbf{M}_n(\mathbb{C})$ containing all scalar matrices. If A has no nontrivial invariant subspaces in $\mathbb{C}^n$, then $A = \mathbf{M}_n(\mathbb{C})$.*

Note that Theorem 1 follows from Theorem 2 by applying the latter to the $\mathbb{C}$-span of the group G. In fact, we see that, in Theorem 1, G could have been replaced by any multiplicative monoid of matrices!

In the standard textbooks I consulted, Theorem 2 is deduced either from Jacobson's Density Theorem ([**7**, p. 648], [**9**, p. 213]), or from its finite-dimensional analogue, Wedderburn's Theorem ([**4**, p. 182], [**8**, p. 109]). These are powerful ring-theoretic results. On the other hand, Theorem 2 is quite elementary in nature; in fact, its statement is completely accessible to an undergraduate class in linear algebra. It seems desirable, therefore, to find a proof of the Theorem using nothing but basic linear algebra techniques.

In the following, we offer such a proof. For the rest of this section, let $V = \mathbb{C}^n$ and $R = \mathbf{M}_n(\mathbb{C})$, and let $A \subseteq R$ be a subring satisfying the hypotheses of Theorem 2.

Lemma 3. *Any $g \in R$ commuting with all $f \in A$ is a scalar matrix.*

Proof: Let $\lambda \in \mathbb{C}$ be an eigenvalue of g, and let $E \subseteq V$ be the associated eigenspace $\{v \in V : gv = \lambda v\}$. For any $f \in A$, $fg = gf$ implies that $f(E) \subseteq E$. Since $E \neq 0$, we have $E = V$, and so $g = \lambda I$. ∎

Lemma 4. *Let $v \in V$ and let W be a subspace of V such that, for any $f \in A$, $f(W) = 0 \Rightarrow f(v) = 0$. Then $v \in W$.*

Proof: We proceed by induction on $\dim W$. The case $\dim W = 0$ is clear, in view of the fact that $I_n \in A$. In case $\dim W > 0$, write W as a sum of a proper subspace W_0 and a line $\mathbb{C}w$ where $w \notin W_0$, and consider the $\mathbb{C}$-subspace

$$H = \{h \in A : h(W_0) = 0\} \subseteq A.$$

By the inductive hypothesis, $H(w) \neq 0$. Since $AH \subseteq H$, we have $A(H(w)) \subseteq H(w)$, and so $H(w) = V$. Now define a linear map $g : V \to V$ by $g(h(w)) = h(v)$ (for any $h \in H$). To check that g is well-defined, suppose $h(w) = 0$ for some $h \in H$. Then $h(W) = 0$, and so $h(v) = 0$ by assumption. Now g commutes with any $f \in A$, since

$$(gf)(h(w)) = g((fh)(w)) = (fh)(v) = f(g(h(w))) = (fg)(h(w))$$

for any $h \in H$. Therefore, by Lemma 3, $g = aI$ for some $a \in \mathbb{C}$. Thus, $h(v) = g(h(w)) = ah(w)$, and so $h(v - aw) = 0$ for any $h \in H$. By the inductive hypothesis again, we have $v - aw \in W_0$, and hence $v \in W$ as desired. ∎

Proof of Theorem 2. It suffices to show that A contains all the matrix units E_{ij}. For ease of notation, assume that $j = 1$. Let $e_1, \dots, e_n \in V$ be the standard basis. Let $H = \{h \in A : h(e_2) = \cdots = h(e_n) = 0\}$. By Lemma 4, $H(e_1) \neq 0$, and as before, $H(e_1)$ is invariant under A. Therefore, $H(e_1) = V$; in particular, there exists $h \in H$ such that $h(e_1) = e_i$. We have then $h = E_{i1} \in A$, as desired. ∎

2. DISCUSSION. (1) In Theorem 1 and Theorem 2, $\mathbb{C}$ could have been replaced by any field k that is algebraically closed.

(2) If the field k is *not* algebraically closed, Theorem 2 is false in general. For instance, if $k = \mathbb{R}$, take the matrix $r = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ in Section 1. Since $r^2 = -I$, $A = kI + kr$ is a subring of $\mathbf{M}_2(k)$. Clearly, A has no nontrivial invariant subspaces in k^2, and yet $A \neq \mathbf{M}_2(k)$.

(3) If k is not algebraically closed, not all is lost. To get Theorem 2 for k, we simply add the assumption that Lemma 3 holds true for A. (This is not an unreasonable assumption, since Lemma 3 is clearly a necessary condition for A to be equal to R.) The proof we gave works *verbatim* in this case.

(4) *In Theorem 2, do we need A to be a subring of R?* A quick look at the proof seems to suggest that A need only be a $\mathbb{C}$-subspace of R containing the identity matrix. However, a closer examination shows that we need A to be a ring in *exactly* one step (used several times), namely, to guarantee the inclusion $AH \subseteq H$. (In ring-theoretic language, H is a left ideal of A.) If A is only a $\mathbb{C}$-subspace (containing I), Theorem 2 need not hold. For instance, take A to be the $\mathbb{C}$-span of I_2 and the matrix units E_{12} and E_{21}. Then $A \neq R$, but, since $E_{11} = E_{12}E_{21}$ and $E_{22} = E_{21}E_{12}$, A has no nontrivial invariant subspaces in $\mathbb{C}^2$. So, if $A \subseteq R$ is a $\mathbb{C}$-subspace of matrices (containing I), the best conclusion is only that A generates R as a ring.

Some beautiful applications of Theorem 1 to the theory of matrix groups are worth mentioning. In **[3]** (which appeared back-to-back with **[2]**), Burnside continued his study of groups of invertible $n \times n$ matrices G over $\mathbb{C}$ and used Theorem 1 to show that, if G has a finite exponent N ($g^N = 1$ for every $g \in G$) or G has a finite number of conjugacy classes, then G is a finite group. With the knowledge of Theorem 1, proofs of these results (suitably generalized to arbitrary fields) are now completely accessible to undergraduates; for an exposition, see [**8**, pp. 151–152]. Apparently, Burnside's results in [**3**] were the origin of the famous Burnside Problems in group theory.

3. PERSPECTIVE. It seems safe to say that a proof such as the one we gave for Theorem 2 does not come from nowhere. Indeed, our proof is closely modeled upon Tate's proof of Wedderburn's Theorem given in Artin's paper [**1**]. Since Tate was proving a more general result, the arguments (and the concomitant notations) given in [**1**] were considerably more involved. Our proof may be viewed as a stripped down version of Tate's proof, written out completely in the elementary language of linear algebra. In particular, our proof made no use of the notions of bimodules, chain conditions, division algebras, etc., and the crucial use of Schur's Lemma (the endomorphism ring of a simple module is a division ring) is snugly hidden behind the eigenspace argument in the proof of Lemma 3.

Tate's proof given in [**1**] does not seem to be as well-known as it should be in linear algebra circles. I wonder if this could be due to the fact that Artin's paper was somehow never reviewed in *Mathematical Reviews*. Perhaps this note will help revive the basic ideas in Tate's proof, and make Theorem 2 into an accessible result in undergraduate linear algebra. A much more sophisticated version of Tate's result formulated as a double-commutant theorem for quasi-injective modules appeared in [**6**].

Note added October 27, 1997. Quite recently, two more proofs of Burnside's Theorem have come to my attention. The first one, by I. Halperin and P. Rosenthal (this MONTHLY **87** (1980) 810) also uses the fact that any linear transformation is a sum of rank 1 transformations. The second one, by E. Rosenthal (*Lin. Algebra Appl.* **63** (1984), 175–177), uses the idea of graph transformations. Some infinite dimensional versions of Burnside's Theorem are available from Chapter 8 of the book *Invariant Subspaces* by H. Radjavi and P. Rosenthal, Springer Verlag, 1973.

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